On the Order of Terms in a Semi-Convergent Series.

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We may give to a semi-convergent series of real terms any value we please by suitably changing the order of its terms. (See Jordan, Cours d'Analyse, t. I, 1893, p. 277.)

Suppose we have a series of the form

$$S = f(1) - f(2) + \dots + f(2n-1) - f(2n) + \dots,$$

where $f(x)$ is a continuous positive function, at least for large values of x.

We have

$$\lim f(n) = 0 \quad \text{and} \quad \lim \frac{f(n+1)}{f(n)} = 1,$$

but

$$f(1) + f(3) + \dots + f(2n-1) + \dots,$$

and

$$f(2) + f(4) + \dots + f(2n) \qquad + \dots,$$

both divergent series.

Suppose for all values of r greater than n

$$\frac{f(r+\theta)}{f(r)} > m \quad \text{and} \quad < M \qquad (0 \leqq \theta \leqq 2),$$

then

$$\int_0^2 f(r+\theta)\, d\theta > \int_0^2 m f(r)\, d\theta$$

and

$$< \int_0^2 M f(r)\, d\theta,$$

or

$$f(r) < \frac{1}{2m} \int_r^{r+2} f(x)\, dx,$$

and

$$> \frac{1}{2M} \int_r^{r+2} f(x)\, dx.$$

m and M are functions of n, and for n indefinitely large they will usually have the same limit, 1.

Now in the series S we will take n' positive terms and n negative terms, $n' > n$, and consider the sum

$$S'_{n'+n} = f(1) + f(3) + \ldots + f(2n'-1) - f(2) - f(4) - \ldots - f(2n),$$

or
$$S'_{n'+n} = S_{2n} + f(2n+1) + f(2n+3) + \ldots + f(2n'-1).$$

[If $n' < n$, these terms will be negative.]

$$\therefore \quad S'_{n'+n} < S_{2n} + \frac{1}{2m} A,$$

and
$$> S_{2n} + \frac{1}{2M} A,$$

where
$$A = \int_{2n+1}^{2n'+1} f(x)\,dx. \qquad (1)$$

There will be a limit if A has a limit, and we shall generally have

$$S' = S + \tfrac{1}{2} \lim A.$$

Since the series S is semi-convergent,

$$\lim \int_a^n f(x)\,dx = \infty, \quad \text{and} \quad \lim \int_n^{n+a} f(x)\,dx = 0,$$

a being a finite quantity.

Equation (1) establishes a relation between n' and n if we assign some value to A; or establishes the value of A corresponding to some relation connecting n' and n.

This relation will reduce approximately for large values of n to a simple form satisfied by integer values of n' and n.

If these solutions are

$$n' = n_1, \ n_3, \ \ldots, \ n_{2r-1}, \ \ldots,$$
$$n = n_2, \ n_4, \ \ldots, \ n_{2r}, \ \ldots,$$

and if we write

$$\alpha_{2r-1} = f(2n_{2r-3}+1) + f(2n_{2r-3}+3) + \ldots + f(2n_{2r-1}-1),$$
$$\alpha_{2r} = f(2n_{2r-2}+2) + f(2n_{2r-2}+4) + \ldots + f(2n_{2r}),$$

we have

$$S' = (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4) + \ldots + (\alpha_{2r-1} - \alpha_{2r}) + \ldots$$
$$= S + \tfrac{1}{2} \lim A.$$

The parentheses may be removed if the α's tend to zero as a limit, and in any case we can arrange the order of the terms in the group $\alpha_{2r-1} - \alpha_{2r}$ in such

a way that the parenthesis will not be necessary. See second example discussed below.

S', then, will be a series whose terms are the terms of the series S arranged in a different order, and we see how we can actually change the order of the terms of a given semi-convergent series, if it can be expressed as assumed above, so that it will take any value we please; and, conversely, we have a method of getting the value of the new series S' produced from S by a given change in the order of the terms, if the change can be expressed by a relation between n' and n as assumed.

Take for example the series

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots .$$

Here (1) becomes

$$\log(2n' + 1) = \log(2n + 1) + A.$$

Putting $A = 2\log a$,

$$2n' + 1 = (2n + 1)a^2,$$

or we may say

$$n' = a^2 n.$$

We shall then get

$$S' = \log(2a).$$

We might have put $A = 2\log[\tfrac{1}{2}f(n)]$,

$$\therefore \quad n' = [\tfrac{1}{2}f(n)]^2 n$$

and

$$S' = \lim \log[f(n)].$$

As another example take

$$S = 1 - \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{2n-1}} - \frac{1}{\sqrt{2n}} + \ldots .$$

(1) becomes

$$\sqrt{2n' + 1} = \sqrt{2n + 1} + \tfrac{1}{2}A.$$

Putting $A = 4a$, and $2n + 1 = (2r + 1)^2$,

$$\begin{aligned} n &= 2r(r + 1), \\ n' &= 2(a + r)(a + r + 1), \\ n' - n &= 2a(a + 2r + 1); \\ \Delta n &= 4(r + 1), \\ \Delta n' &= 4(a + r + 1), \\ \Delta^2 n &= \Delta^2 n' = 4. \end{aligned}$$

Hence we have have the following system of values:

$$
\begin{array}{rlllll}
\text{For } r = & 1, & 2, & 3, \ldots\ldots r & , \ldots\ldots, \\
\Delta n = 4, & 8, & 12, & \ldots\ldots & 4(r+1), \ldots\ldots, \\
n = & 4, & 12, & 24, \ldots\ldots & 2r(r+1), \ldots\ldots,
\end{array}
$$

$$\alpha_{2r} = \frac{1}{\sqrt{4r(r-1)+2}} + \ldots\ldots + \frac{1}{\sqrt{4r(r+1)}}.$$

$$
\begin{array}{rllll}
\Delta n' = 4(a+1), & 4(a+2), & 4(a+3), \ldots\ldots & 4(a+r+1) & , \ldots\ldots, \\
n' = & 2(a+1)(a+2), & 2(a+2)(a+3) & , \ldots\ldots 2(a+r)(a+r+1), \ldots\ldots,
\end{array}
$$

$$\alpha_{2r-1} = \frac{1}{\sqrt{4(a+r)(a+r-1)+1}} + \ldots\ldots + \frac{1}{\sqrt{4(a+r)(a+r+1)-1}}.$$

$$2\sqrt{4(a+r)(a+r+1)+1} = 2(2a+2r+1),$$

$$2\sqrt{4r(r+1)+2} = 4r + 2 + \frac{1}{2r} - \frac{1}{4r^2} + \ldots\ldots$$

An application of the method employed in this paper will give

$$\lim \alpha_{2r-1} = \lim\left[2(2a+2r+1) - 2(2a+2r-1)\right] = 4,$$

$$\lim \alpha_{2r} = \lim\left[2\sqrt{4r(r+1)+2} - 2\sqrt{4r(r-1)+2}\right]$$

$$= \lim\left(4 - \frac{1}{2r^2} + \ldots\ldots\right),$$

therefore

$$\lim(\alpha_{2r-1} - \alpha_{2r}) = \lim\left(\frac{1}{2r^2} - \ldots\ldots\right).$$

The series

$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \ldots\ldots$$

in this case will not be convergent but "oscillating." We may, however, mix the positive and negative terms, as suggested above, in such a way as to produce a convergent series having for value $S + 2a$ without using any parentheses.

α_{2r-1} and α_{2r} consist of terms of the original series $\Delta n'$ and Δn in number. Our rule then would be: Take $a+r+1$ positive terms and distribute among them $r+1$ negative terms. Do this four times for every value of r, beginning with zero.

Suppose we take $a = -1$. We have

$$S' = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{8}}$$

$$-\frac{1}{\sqrt{10}} + 1 - \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{16}} - \frac{1}{\sqrt{18}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{20}} - \frac{1}{\sqrt{22}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{24}}$$

$$- \ldots . = S - 2.$$

We can say in regard to the series S: If $n' - n$ is a quantity of the same order of magnitude as n, S' will be divergent. If $n' - n$ is finite, $S' = S$. This will be true of any semi-convergent series.

If $n' - n$ is of the same order of magnitude as the square root of n, S' will usually converge to a value different from S.

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